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Chiral Potts Model with Skewed Boundary Conditions

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We obtain the transfer matrix functional relations for the chiral Potts model with skewed boundary conditions and find that they are the same as for periodic boundary conditions, but with modified selection rules. As a start toward calculating the interfacial tension in general, we here evaluate it in a low-temperature limit, performing a Bethe-ansatz-type calculation. Finally, we specialize the relations to the superintegrable case, verifying the ansatz proposed by Albertini *et al.*

KEY WORDS: Statistical mechanics; lattice models; chiral Potts model; wetting; Bethe ansatz.

1. INTRODUCTION

The "chiral Potts model" is a planar lattice model with N-state spins that live on the sites of the lattice and interact along edges. The interactions are chosen so that the star-triangle relations[(1)] are satisfied, and because of these we expect the model to be "solvable," in the sense that we should be able to calculate the bulk free energy and some other large-lattice properties, such as the correlation length and interfacial tension.

The special "superintegrable" case has been extensively studied, particularly when $N=3$.[(2–14)] Much less is known for the general solvable model, but for $N=3$ Albertini *et al.* have postulated functional relations satisfied by the row-to-row transfer matrices.[(4)] These and other relations have been derived for general N by Bazhanov and Stroganov[(15)] and by Baxter *et al.*[(16)] All these relations assume periodic (nonskewed) boundary conditions: if each row of the lattice contains L sites, carrying spins

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$\sigma_1,..., \sigma_L$, (ordered from left to right), then the spin to the right of σ_L is σ_{L+1}, where $\sigma_{L+1} = \sigma_1$.

These relations also define the eigenvalues of the transfer matrices. The bulk free energy can be obtained from the maximum eigenvalue and has been derived as a double integral.[17,18]

However, as yet no calculations appear to have been made of the correlation length and interfacial tension for the general solvable model (these are obtainable from the next-to-largest eigenvalues, or from the largest eigenvalue with skewed boundary conditions). Here we make a start in this direction by deriving the functional relations for the skewed boundary conditions $\sigma_{L+1} = \sigma_1 - r$, where r is some fixed integer (the "skew parameter") which we can choose to be between 0 and $N-1$.

We find an intriguing property, namely that the only change in the functional relations is to replace the quantum number Q of the spin-shift operator by $Q+r$, module N (see Section 3). Thus we obtain the same solution set as before (when $r=0$), but the selection rules are different. This must account for at least some of the "spurious solutions" that have previously been reported.[19]

Having obtained the functional relations, in Section 4 we go on to put them into an explicitly real form. Then, in order to locate the desired solutions corresponding to the largest eigenvalues, in Section 5 we consider a zero-temperature limit. Because of the skewed boundary conditions, this turns out to be nontrivial, involving a Bethe ansatz (or functional relation) calculation for r dislocations or "particles" moving through the lattice.

This calculation supports a Z-invariance argument that the vertical interfacial tension (as here defined) should be independent of the vertical rapidities p', p'. In Section 6 we therefore consider the particular choice of these rapidities that makes the model "superintegrable" and show how the functional relations simplify for this case (still with skewed boundary conditions). We verify the "ansatz" used for periodic boundary conditions by Albertini *et al.*[5]

It should be noted that the zero-temperature limit considered here is different from that mentioned after Eq. (3.52) of ref. 20 and in ref. 21, where the relationship of the integrable chiral Potts model to the wetting line of the Ostlund–Huse model is discussed. This is because the interfacial tension of this model is orientation dependent, and the orientation considered in refs. 20 and 21 is different from that used here. Our choice is made to (a) ensure independence of one set of rapidities, and (b) make the zero-temperature problem solvable by a simple Bethe ansatz. We hope to go on in a subsequent publication to study nonzero temperatures: then it should be possible to consider the effect of 90° rotations of the lattice.

2. BOLTZMANN WEIGHTS AND THE TRANSFER MATRIX

We define the chiral Potts model in the usual way.[1,17] Consider the square lattice $\mathscr{L}$, drawn diagonally as in Fig. 1, with L sites per row. At each site i there is a spin σ_i, which takes values $0,..., N-1$. There is an associated lattice $\mathscr{L}'$ denoted by dotted lines, such that each edge of $\mathscr{L}$ passes through a vertex of $\mathscr{L}'$.

Let $\omega = e^{2\pi i/N}$ be the primitive Nth root of unity and let k and k' be two real positive parameters satisfying

$$k^2 + k'^2 = 1 \tag{2.1}$$

Also, let $q = \{x_q, y_q, \mu_q\}$ be a set of complex parameters ("q-variables"), related by

$$x_q^N + y_q^N = k(1 + x_q^N y_q^N), \qquad kx_q^N = 1 - k'\mu_q^{-N}, \qquad ky_q^N = 1 - k'\mu_q^N \tag{2.2}$$

and further define

$$t_q = x_q y_q, \qquad \Lambda_q = \mu_q^N \tag{2.3}$$

Only one of these variables is independent. In terms of the a_q, b_q, c_q, and d_q of ref. 1, $x_q = a_q/d_q$, $y_q = b_q/c_q$, and $\mu_q = d_q/c_q$. We refer to q as a "rapidity."

Similarly, define "p-variables" $p = \{x_p, y_p, \mu_p\}$, and t_p. To each vertical (horizontal) dotted line of Fig. 1 assign a rapidity p (q). In general they may be different for different lines. In fact a convenient level of generality that we shall use here is to allow the vertical rapidities to be alternately $p, p', p, p',...$, as indicated. Then on a SW $\to$ NE edge (i, j) of $\mathscr{L}$ (with j

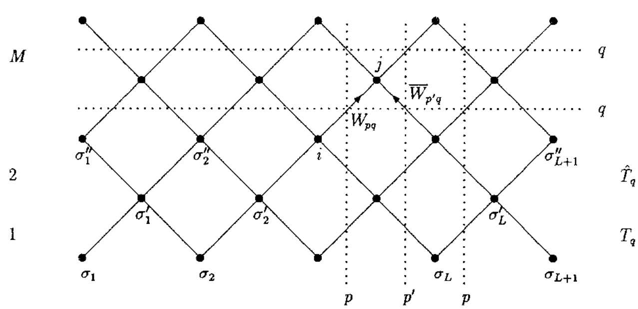

Fig. 1. The square lattice $\mathscr{L}$ of M rows with L sites per row. T_q is the transfer matrix of an odd row, $\hat{T}_q$ of an even row. Three vertical and two horizontal dotted rapidity lines are shown.

above i), the spins σ_i, σ_j interact with Boltzmann weight $W_{pq}(\sigma_i-\sigma_j)$ [or $W_{p'q}(\sigma_i-\sigma_j)$], where (for all integers n)

$$W_{pq}(n)=(\mu_p/\mu_q)^n \prod_{j=1}^{n} (y_q-\omega^j x_p)/(y_p-\omega^j x_q) \tag{2.4}$$

Similarly, on SE $\to$ NW edges the spins interact with Boltzmann weight $\bar{W}_{pq}(\sigma_i-\sigma_j)$, where

$$\bar{W}_{pq}(n)=(\mu_p\mu_q)^n \prod_{j=1}^{n} (\omega x_p-\omega^j x_q)/(y_q-\omega^j y_p) \tag{2.5}$$

Here we normalize so that $W_{pq}(0)=\bar{W}_{pq}(0)=1$. The weights satisfy the periodicity conditions $W_{pq}(n+N)=W_{pq}(n)$, $\bar{W}_{pq}(n+N)=\bar{W}_{pq}(n)$. They also satisfy the star–triangle relation[1,20,22]:

$$\begin{aligned}\sum_{d=0}^{N-1} \bar{W}_{qr}(b-d)\, W_{pr}(a-d)\, \bar{W}_{pq}(d-c)\\ =(f_{pq}f_{qr}/f_{pr})\, W_{pq}(a-b)\, \bar{W}_{pr}(b-c)\, W_{qr}(a-c)\end{aligned} \tag{2.6}$$

for all rapidities p, q, r and all integers (spins) a, b, c. Here f_{pq} is a spin-independent function, defined by

$$f_{pq}^N=\det{}_N[\bar{W}_{pq}(i-j)]\Big/\prod_{n=0}^{N-1} W_{pq}(n)$$

It can be written in product form by using the identity (2.44) of ref. 16, namely

$$\det{}_N[\bar{W}_{pq}(i-j)]=N^{N/2}e^{i\pi(N-1)(N-2)/12}\prod_{j=1}^{N-1}\frac{(t_p-\omega^j t_q)^j}{(x_p-\omega^j x_q)^j\,(y_p-\omega^j y_q)^j} \tag{2.7}$$

We define row-to-row transfer matrices T and $\hat{T}$ as in ref. 16. Let $\sigma=\sigma_1,\dots,\sigma_L$ be the spins in the lower row of Fig. 1. Similarly, let $\sigma'=\sigma'_1,\dots,\sigma'_L$ be the spins in the next row, and $\sigma''=\sigma''_1,\dots,\sigma''_L$ those in the row above that. Let T be the N^L by N^L matrix with elements

$$T_{\sigma\sigma'}=\prod_{J=1}^{L} W_{pq}(\sigma_J-\sigma'_J)\, \bar{W}_{p'q}(\sigma_{J+1}-\sigma'_J) \tag{2.8}$$

Similarly, let $\hat{T}$ be the N^L matrix with elements

$$\hat{T}_{\sigma'\sigma''}=\prod_{J=1}^{L} \bar{W}_{pq}(\sigma'_J-\sigma''_J)\, W_{p'q}(\sigma'_J-\sigma''_{J+1}) \tag{2.9}$$

In ref. 16 we used the periodic boundary conditions $\sigma_{L+1}=\sigma_1$, $\sigma''_{L+1}=\sigma''_1$. Here we generalize these to the "skewed" boundary conditions

$$\sigma_{L+1}=\sigma_1-r, \qquad \sigma''_{L+1}=\sigma''_1-r, \quad \text{mod } N \tag{2.10}$$

where r is any integer (without loss of generality we can restrict it to the range $0 \leqslant r < N-1$). We know that in the Ising case such conditions provide a natural completion of the eigenvalue spectrum. Also, if we can obtain the maximum eigenvalue of $T\hat{T}$ for arbitrary r, then we can deduce the interfacial tensor.[23,24]

The matrices T and $\hat{T}$ are the row-to-row transfer matrices of the model (see, for comparison, Chapter 7 of ref. 25). If the lattice $\mathscr{L}$ has $2M$ rows (and hence $2LM$ sites), then the partition function is

$$Z=\text{Trace } T\hat{T}T\hat{T}\cdots\hat{T}=\text{Trace}(T\hat{T})^M \tag{2.11}$$

there being one transfer matrix T or $\hat{T}$ for each row.

We can regard the vertical rapidities p and p' as fixed constants, the horizontal rapidity q as a variable, and denote T, $\hat{T}$ explicitly as T_q, $\hat{T}_q$. Then for periodic boundary conditions, it is shown in (2.32) of ref. 16 that the transfer matrices satisfy the commutation property,

$$T_q\hat{T}_r=(f_{p'q}f_{pr}/f_{pq}f_{p'r})^L T_r\hat{T}_q \tag{2.12}$$

together with a corresponding property obtained by interchanging T with $\hat{T}$, and p with p'. These properties are direct results of the star–triangle (or "Yang–Baxter") relation, and hold for all horizontal rapidities q and r. Their derivation extends at once [because it remains true that $W_{qr}(\sigma_{L+1}-\sigma''_{L+1})=W_{qr}(\sigma_1-\sigma''_1)$] to the skewed boundary conditions (2.10).

It follows that the matrices T_q and $\hat{T}_q$ can be simultaneously diagonalized by the coupled similarity transformations $T_q\to P^{-1}T_qQ$ and $\hat{T}_q\to Q^{-1}\hat{T}_qP$, where P and Q are constant matrices, independent of q. Throughout this paper, by "eigenvalues" and "eigenvectors" of T_q, $\hat{T}_q$ we mean the solutions of the coupled vector equations

$$T_q\mathbf{y}=(\text{scalar})\mathbf{x}, \qquad \hat{T}_q\mathbf{x}=(\text{scalar})\mathbf{y} \tag{2.13}$$

$\mathbf{x}$ and $\mathbf{y}$ are the eigenvectors (independent of q), the scalars are the eigenvalues of T_q, $\hat{T}_q$, $\tau_j(t)$. Postmultiplying any matrix relation by the appropriate $\mathbf{x}$ or $\mathbf{y}$ [e.g., (2.12) by $\mathbf{x}$] effectively replaces each matrix by its eigenvalue. For this reason we shall often use the same notation for the eigenvalues as for the matrices themselves. For instance, (2.12) can be regarded as a relation either between matrices or between their corresponding eigenvalues.

In ref. 16 we also introduced a related set of N^L by N^L matrices $\tau_{k,q}^{(j)}$ for $k=0,\dots, N-1$ and $j=0,\dots, N$. Here we shall regard these as defined by the functional relations of the next section, but for completeness we also given the direct definition of $\tau_{k,q}^{(2)}$. For all integers a, b, c, d, let $\alpha=a-d+k$ (mod N) and $\beta=b-c+k$ (mod N). Define a function $F_{pq}(\alpha, n)$ by

$$\begin{aligned} F_{pq}(0,0) &= y_p, \qquad F_{pq}(0,1) = -\omega t_q \\ F_{pq}(1,0) &= \mu_p, \qquad F_{qq}(1,1) = -\omega x_p \mu_p \\ F_{pq}(\alpha, n) &= 0 \qquad \text{if} \quad \alpha \neq 0 \quad \text{or} \quad 1 \end{aligned} \tag{2.14}$$

and hence define

$$U(a, d \mid b, c) = \sum_{n=0}^{1} \omega^{n(d-b-k)}(-\omega t_q)^{\alpha-n} F_{pq}(\alpha, n) F_{p'q}(\beta, n) \tag{2.15}$$

Then

$$\{\tau_{k,q}^{(2)}\}_{\sigma\sigma''} = \prod_{J=1}^{L} U(\sigma_J, \sigma_J'' \mid \sigma_{J+1}, \sigma_{J+1}'') \tag{2.16}$$

Nopte that an element of $\tau_{k,q}^{(2)}$ is zero unless $\sigma_J - \sigma_J'' + k = 0$ or 1 (mod N) for all $J=1,\dots, L$. Also, each element of $\tau_{k,q}^{(2)}$ is a polynomial in t_q of degree at most L. To simplify the equations, we have multiplied the matrices $\tau_{k,q}^{(j)}$ of ref. 16 by $(y_p y_{p'})^{(j-1)L}$ and the function $z(t)$ by $(y_p y_{p'})^{2L}$.

It is shown in ref. 16 that the matrices $\tau_j(t)$ commute with $T_q \hat{T}_r$ (for all t, q, r), so they can also be diagonalized by the similarity transformation $\tau_j(t) \to P^{-1}\tau_j(t)P$ and have the vector $\mathbf{x}$ of (2.13) as an eigenvector.

As is remarked in ref. 16 [after eq. (3.44)], we can think of $\tau_{k,q}^{(2)}$ as the transfer matrix of a mixed model in which there are chiral Potts spins on alternate rows of $\mathscr{L}$, with new "n-spins" in between that take only the values 0 and 1. The τ-matrices are related to the column-to-column transfer matrices of the "superintegrable" chiral Potts model[2-14]: if we impose fixed-spin boundary conditions, then they each have a very simple direct-product eigenvalue spectrum (see Section 6 of ref. 16, and refs. 6, 26, and 27). Note, however, that we do not do this here: we use the skewed boundary conditions (2.10).

Spin-Shift Symmetry. The model is Z_N invariant, i.e., it is unchanged by incrementing every spin σ_i by one (mod N). Let X be the spin-shift operator, with elements

$$X_{\sigma\sigma'} = \prod_{J=1}^{L} \delta(\sigma_J, \sigma_J' + 1) \tag{2.17}$$

taking $\delta(a, b) = 1$ if $a = b$ (mod N), else $\delta(a, b) = 0$. Then $X^N = 1$ and X commutes with all the T and τ transfer matrices, in particular

$$XT_q = T_q X, \qquad X\hat{T}_q = \hat{T}_q X \tag{2.18}$$

For the eigenvalue equations we can therefore replace X by its eigenvalue, setting

$$X = \omega^Q \tag{2.19}$$

where $Q = 0, \dots, N-1$.

Further, let V be the automorphism (called T in refs 1 and 16) such that

$$x_{Vp} = \omega x_p, \qquad y_{Vp} = \omega^{-1} y_p, \qquad \mu_{Vp} = \omega^{-1}\mu_p \tag{2.20}$$

Then

$$W_{p,Vq}(n) = \frac{W_{pq}(n+1)}{W_{pq}(1)}, \qquad \bar{W}_{p,Vq}(n) = \frac{\bar{W}_{pq}(n+1)}{\bar{W}_{pq}(1)} \tag{2.21}$$

It follows that

$$\begin{aligned} T_{Vq} &= [W_{pq}(1)\, \bar{W}_{p'q}(1)]^{-L} X^{-1} T_q \\ \hat{T}_{Vq} &= [W_{p'q}(1)\, \bar{W}_{pq}(1)]^{-L} X^{-1} \hat{T}_q \end{aligned} \tag{2.22}$$

For all complex numbers x, y and all integers m, n, we introduce the notation

$$\begin{aligned} (x, y)_{m,n} &= \prod_{j=m+1}^{n} (x - y\omega^j), \qquad && n \geqslant m \\ &= \prod_{j=n+1}^{m} (x - y\omega^j)^{-1}, \qquad && n \leqslant m \end{aligned} \tag{2.23}$$

Using this, we define, for all integers k, l, j such that $j = k + l$,

$$\Lambda_q^{(k,l)} = \left[\frac{(y_p, x_q)_{0,l-1} (y_q, x_p)_{-k,0} (y_q, y_{p'})_{0,N-k-1}}{N\mu_p^{-k} \mu_{p'}^{l} (x_{p'}, x_q)_{-1,l-1}} \right]^L \tag{2.24}$$

$$\begin{aligned} H_{pq}^{(j)} &= [\omega^j \mu_p^j (t_p, t_q)_{-1,j-1} / (y_p^N - x_q^N)]^L \\ \bar{H}_{p',q}^{(j)} &= [\mu_{p'}^{-j} (t_{p'}, t_q)_{j-1,N-1} / (x_{p'}^N - x_q^N)]^L \end{aligned} \tag{2.25}$$

These are the $\lambda_q^{(k,l)}$, $H_{pq}^{(j)}$, $\bar{H}_{p'q}^{(j)}$ used in Section 3 of ref. 16, except that we have multiplied each by $(-\mu_q)^{jL} \omega^{-j(j-1)L/2} y_p^{(j-1)} y_{p'}^{(N-j-1)L}$, and $H_{pq}^{(j)}$ by a further $(y_p y_{p'})^{(1+j-N))L}$, and $\bar{H}_{pq}^{(j)}$ by $(y_p y_{p'})^{(1-i)L}$.

The set of variables $q=\{x_q, y_q, \mu_q\}$ defines the Boltzmann weights and the transfer matrices (considered as functions of q) uniquely. Let $\bar{q}(k, l)$, or simply $\bar{q}kl$, be the related set $\{\omega^k y_q, \omega^l x_q, \mu_q^{-1}\}$. Define

$$S_{q|kl}=\omega^{rk}\Lambda_q^{(k,l)}X^k\hat{T}_{\bar{q}kl} \tag{2.26}$$

Substituting the forms (2.4), (2.5) into (2.8), we find that μ_q enters the RHS only via the factor $\mu_q^{\sigma_{L+1}-\sigma_1}=\mu_q^{-r}$. (Apart possibly from factors of μ_q^N that can arise from interpreting the spin differences to modulo N.) Hence $\mu_q^r T_q$ is a single-valued rational function of x_q and y_q. Similarly, so is $\mu_q^r\hat{T}_q$. Using this fact, replacing q in (2.22) by $\bar{q}kl$, we obtain

$$S_{q|k+1,l-1}=S_{q|kl} \tag{2.27}$$

Hence $S_{q|kl}$ depends on k and l only via their sum $j=k+l$, and we can write it simply as $S_q^{(j)}$.

3. FUNCTIONAL RELATIONS

3.1. The $T\hat{T}$, $\tau^{(j)}$ Relations

We now present the generalization to skewed boundary conditions of the transfer matrix functional relations given in ref. 16. Let us denote equations of that paper with the prefix BBP.

The derivation of (BBP3.46) goes through as before, except for the remark two lines before Eq. (BBP3.41). The factors involving ω^{-kd} and ω^{kc} now give a contribution $\omega^{k(\sigma''_{J+1}-\sigma''_J)}$ to the prodand in (BBP2.27), and hence contribute an extra factor $\omega^{k(\sigma''_{L+1}-\sigma''_1)}=\omega^{-kr}$ to the overall product, i.e., to the first term on the RHS of (BBP3.46). Corresponding extra factors arise in the second term, given a contribution $\omega^{l(\sigma''_J-\sigma''_{J+1})}$ to the prodand, and hence an overall factor ω^{lr}. Using our definition (2.26), we see that (BBP3.46) therefore generalizes to

$$\omega^{-rk}X^{-k}T_q S_q^{(j)}=\omega^{-rk}\bar{H}_{p'q}^{(j)}\tau_{k,q}^{(j)}+\omega^{lr}H_{pq}^{(j)}\tau_{k-j,\bar{q}kl}^{(N-j)} \tag{3.1}$$

true for any rapidity q and all integers k, l, j satisfying $k+l=j$, where $0\leqslant j\leqslant N$. As before,

$$\tau_{k,q}^{(0)}=0, \qquad \tau_{k,q}^{(1)}=X^{-k} \tag{3.2}$$

$\tau_{k,q}^{(j)}$ depends on q simply as a polynomial in t_q of degree $(j-1)L$, and for $j=0,\ldots, N$

$$X\tau_{k,q}^{(j)}=\tau_{k,q}^{(j)}X=\tau_{k-1,q}^{(j)} \tag{3.3}$$

We can use these properties of $\tau_{k,q}^{(j)}$ to simplify our notation, writing

$$\tau_{k,q}^{(j)} = X^{-k}\tau_j(t_q) \tag{3.4}$$

where $\tau_0(t) = 0$, $\tau_1(t) = 1$.

Multiplying (3.1) by $\omega^{rk}X^k$, we find that it simplifies to

$$T_q S_q^{(j)} = \bar{H}_{p'q}^{(j)}\tau_j(t_q) + \omega^{jr}X^j H_{pq}^{(j)}\tau_{N-j}(\omega^j t_q) \tag{3.5}$$

which makes it clear that there is only one equation for each value of j.

We could have obtained this result another way: there is a simple trick [mentioned before Eq. (3.46) of ref. 20] whereby one can generalize the functional relations of ref. 16 to skewed boundary conditions. First note that the equations generalize at once to the fuly column-inhomogeneous case, when the $2L$ vertical rapidities can differ arbitrarily from column to column. This point is discussed before (BBP4.48). In the functional relations, the scalar coefficients are products of expressions of the form $f(p)^L$ and $g(p')$, where $f(p)$ and $g(p')$ are some functions. To generalize, all one has to do is to replace such expressions by the products $f(p_1)f(p_3)\cdots f(p_{2L-1})$ and $g(p_2)g(p_4)\cdots g(p_{2L})$, p_{2L-1} and p_{2L} being the rapidities of the p- and p'-type lines (respectively) between sites L and $L+1$ in the lowest row of Fig. 1.

Suppose we do this, and then take $p_1, p_2, \ldots, p_{2L}$ to be $p, p', p, \ldots, p', p, Vp'$, where V is the automorphism defined in (2.20). Thus we only change the last vertical rapidity, taking it from p' to Vp'. From (2.4) and (2.5),

$$W_{Vp',q}(n) = \frac{W_{p'q}(n+1)}{W_{p'q}(1)}, \qquad \bar{W}_{Vp',q}(n) = \frac{\bar{W}_{p'q}(n-1)}{\bar{W}_{p'q}(N-1)} \tag{3.6}$$

We make these substitutions in (2.8) and (2.9), using the periodic boundary conditions of ref. 16. Only the very last terms in the products are affected, giving

$$\begin{aligned} T &= [\bar{W}_{p'q}(N-1)]^{-1}T^* \\ \hat{T} &= [W_{p'q}(1)]^{-1}\hat{T}^* \end{aligned} \tag{3.7}$$

where T^* and $\hat{T}^*$ are the transfer matrices of this paper, *with the skewed boundary conditions* $\sigma_{L+1} = \sigma_1 - 1$, $\sigma''_{L+1} = \sigma''_1 - 1$, i.e., with $r = 1$ in (2.10). Further, the function F defined in (BBP3.38) satisfies

$$F_{Vp',q}(j, \alpha, n) = \omega^n F_{p',q}(j, \alpha, n) \tag{3.8}$$

and from (BBP3.44a) it follows that the τ-matrices of ref. 16 (with $r=0$) become those of this paper, with $r=1$.

Making these substitutions in (BBP3.46), we do indeed obtain our result (3.5) for the case of skewed boundary conditions with $r=1$. Iterating, we obtain it for arbitrary values of r.

3.2. The Other Relations

The functional relations (BBP4.20) and (BBP4.21) are unchanged by the substitution $p' \to Vp'$ in the last column, so are true for all values of the skew parameter r. In particular, using our matrix $S_q^{(j)}$, we can write (BBP4.21)

$$S_q^{(j+1)}\tau_2(\omega^j t_q) = \omega^r[\omega\mu_p(t_p - \omega^j t_q)]^L XS_q^{(j)} + [\mu_{p'}(t_{p'} - \omega^{j+1}t_q)]^L S_q^{(j+2)} \tag{3.9}$$

This equation is true for all integers j; $S_q^{(j)}$ is not fully periodic of period N, rather it satisfies the quasiperiodicity relation

$$S_q^{(j+N)} = \left[\frac{(y_p^N - x_q^N)}{\mu_{p'}^N(x_{p'}^N - x_q^N)}\right]^L S_q^{(j)} = \left(\frac{1-\mu_p^N\mu_q^N}{\mu_{p'}^N - \mu_q^N}\right)^L S_q^{(j)} \tag{3.10}$$

The only effect of the $p' \to Vp'$ substitution on the τ-relations (BBP4.27) is to multiply each function z by a factor ω. Iterating, this becomes a factor ω^r and the equations become, for any integer value of r,

$$\begin{aligned} \tau_j(t)\,\tau_2(\omega^{j-1}t) &= \omega^r Xz(\omega^{j-1}t)\,\tau_{j-1}(t) + \tau_{j+1}(t) \\ \tau_j(\omega t)\,\tau_2(t) &= \omega^r Xz(\omega t)\,\tau_{j-1}(\omega^2 t) + \tau_{j+1}(t) \\ \tau_{N+1}(t) &= \omega^r Xz(t)\,\tau_{N-1}(\omega t) + \alpha_q + \bar{\alpha}_q \end{aligned} \tag{3.11}$$

where the first two sets of relations are equivalent, holding for $j=1,\ldots, N$, and

$$z(t_q) = [\omega\mu_p\mu_{p'}(t_p - t_q)(t_{p'} - t_q)]^L \tag{3.12}$$

$$\alpha_q = [\mu_q^N(y_p^N - x_q^N)(y_{p'}^N - x_q^N)/k']^L \tag{3.13}$$

$$\bar{\alpha}_q = [\mu_q^{-N}(y_p^N - y_q^N)(y_{p'}^N - y_q^N)/k']^L$$

We remark again that throughout this paper x_p, $x_{p'}$, y_p, $y_{p'}$, μ_p, $\mu_{p'}$ are regarded as fixed constants; and x_q, y_q, μ_q as complex variables, satisfying the relations (2.2).

Using (2.2), we can establish that

$$z(t_q)\, z(\omega t_q) \cdots z(\omega^{N-1} t_q) = \alpha_q \bar{\alpha}_q \tag{3.14}$$

To summarize so far: we have generalized (and slightly simplified) the functional matrix relations of ref. 16 to skewed boundary conditions, the resulting relations being (3.5), (3.9), (3.11). They are not independent: as is expained in ref. 16, (3.11) can be obtained by premultiplying (3.9) by T_q, then using (3.5) and considering the effect of interchanging x_q with y_q.

Postmultiplying (3.5), (3.9), (3.11) by $\mathbf{x}$, it becomes obvious that the functional relations continue to hold if the matrices are replaced by their eigenvalues (there are many solutions, corresponding to the various eigenvalues): we shall use the same notation both for the matrices and their eigenvalues.

Note that r and X enter these functional matrix relations only via the combination $\omega^r X$, so in fact the equations are the same as for periodic boundary conditions, but with a modified interpretation of X.

Using (2.19), we can replace by its appropriate eigenvalue ω^Q, where $Q = 0, \ldots, N-1$ (i.e., Q is a quantum number for the equations), so Q is simply replaced by $Q + r$ (mod N). Otherwise the equations are unaltered and we have the same solution set as we had before (when r was zero). The only difference is that we have to select a new subset of these as the allowed solutions for a given r. Precisely this happens in the Ising model, where for skewed boundary conditions ($\sigma_{L+1} \neq \sigma_1$) we still obtain the result (3.38) of ref. 28 for the eigenvalues of the transfer matrix, but the accompanying selection rules (the number of ξ-particles being even or odd) are reversed.

Another (related) functional relation that we shall need in (BBP4.20). If we explicitly manifest T_q as a single-valued function of x_q, y_q, μ_q and write it as $T(x_q, y_q, \mu_q)$, then by using (2.22) and the property mentioned before (2.27), we can write (BBP4.20) (in the normalization of this paper) as

$$\begin{aligned}\tau_2(t_q)\, T(x_q, \omega y_q, \mu_q) = \omega^r X &\left[\frac{\omega \mu_p \mu_{p'} (x_p - y_q)(t_{p'} - t_q)}{y_{p'} - y_q}\right]^L T(x_q, y_q, \mu_q) \\ &+ \left[\frac{(y_{p'} - \omega y_q)(t_p - \omega t_q)}{x_p - \omega y_q}\right]^L T(x_q, \omega^2 y_q, \mu_q)\end{aligned} \tag{3.15}$$

We shall find this form convenient for discussing the low-temperature limit.

4. REAL FORMS

The above notation has the advantage of displaying the algebraic form of the functions and coefficients, but obscures that fact that under certain conditions the Boltzmann weights and transfer matrices are real. We introduce constants λ, ζ such that

$$\lambda = \pi/N, \qquad \zeta = e^{i\lambda/2}, \qquad \omega = \zeta^4, \qquad i = \zeta^N \tag{4.1}$$

and further rapidity variables θ_p, θ_p, u_p, v_p, η_p so that

$$\begin{gathered} x_p = e^{2i\phi_p}, \qquad y_p = e^{2i\theta_p} \\ 2\theta_p = u_p + v_p, \qquad 2\phi_p = u_p - v_p, \qquad \mu_p = \zeta^{-1} e^{iv_p} \eta_p \end{gathered} \tag{4.2}$$

Then (2.2) implies that

$$\cos Nv_p = k \cos Nu_p, \qquad \eta_p^N = \frac{\sin 2N\theta_p}{k' \cos Nu_p} = -\frac{k' \cos Nu_p}{\sin 2N\phi_p}, \qquad t_p = e^{2iu_p} \tag{4.3}$$

We have changed the notation slighly from the introduced in §3 of ref. 29 and used in refs. 17 and 18: the θ, ϕ, u, v of those papers have been replaced by $2\theta - \pi/N$, 2ϕ, $Nu - \pi/2$, $Nv - \pi/2$, respectively. We can think of u_p as the fundamental variable, v_p, θ_p, ϕ_p, x_p, y_p, η_p, μ_p being defined by these equations.

Then (2.4), (2.5) can be written

$$\begin{aligned} W_{pq}(n) &= \left(\frac{\eta_p}{\eta_q}\right)^n \prod_{j=1}^{n} \frac{\sin(\phi_p - \theta_q + j\lambda)}{\sin(\phi_q - \theta_p + j\lambda)} \\ \bar{W}_{pq}(n) &= (\eta_p \eta_q)^n \prod_{j=1}^{n} \frac{\sin(\phi_q - \phi_p + j\lambda - \lambda)}{\sin(\theta_p - \theta_q + j\lambda)} \end{aligned} \tag{4.4}$$

and we can deduce that

$$\bar{W}_{pq}(-n) = (\eta_p \eta_q)^{-n} \prod_{j=1}^{n} \frac{\sin(\theta_q - \theta_p + j\lambda - \lambda)}{\sin(\phi_p - \phi_q + j\lambda)} \tag{4.5}$$

Remembering that $0 < k < 1$, we see that if u_p, u_q are real, then so are v_p, v_q, η_p^N, η_q^N. If v_p and v_q are chosen between 0 and λ, then η_p^N and η_q^N are positive and we can choose η_p and η_q to be positive real, so that Boltzmann weights $W_{pq}(n)$, $\bar{W}_{pq}(n)$ are real. Further, if $u_p < u_q < u_p + \lambda$, then the Boltzmann weights are positive, as is required for a physical syustem. From the Perron–Frobenius theorem (for all values of the skew parameter r), the maximum eigenvalues of T_q and $\hat{T}_q$ must then be positive real, as must be the elements of the eigenvectors. Hence the corresponding eigenvalues of T_q and $\hat{T}_q$ must be real, for *all* q with u_q, η_q real and $0 < v_q < \lambda$.

The same is not necessarily true for $\hat{T}_{\bar{q}kl}$. Remembering that $x_{\bar{q}kl} = \omega^k y_q$, $y_{qpkl} = \omega^l x_q$, $\mu_{\bar{q}kl} = 1/\mu_q$, we see that $u_{\bar{q}kl} = u_q + (k + l)\lambda$,

$v_{\bar{q}kl} = -v_q + (l-k)\lambda$, $\eta_{\bar{q}kl} = \zeta^{2(k-l+1)}/\eta_q$. Hence $\eta_{\bar{q}kl}$ is not in general real. However, its Nth power is real, so from the remarks before (2.27)

$$\hat{T}_{\bar{q}kl} = \zeta^{2r(l-k-1)} \times \{\text{real matrix}\} \tag{4.6}$$

Let

$$\begin{aligned} \beta_q &= (2\zeta)^{-L} \exp[-iL(\theta_q + \phi_q + \theta_p + \theta_{p'})] \\ \gamma_q^{(j)} &= 2^{-L(N-1)} \zeta^{-L(N+j+1)} \exp\{-iL[N\theta_q + j\theta_p + (N-j)\theta_{p'}]\} \end{aligned} \tag{4.7}$$

and define modified coefficients and matrices

$$\begin{aligned} A_q^{(k,l)\dagger} &= \beta_q^{-1} \gamma_q^{(k+l)} A_q^{(k,l)} \\ H_{pq}^{(j)\dagger} &= \zeta^{L(N+1-j)(j+1)} \beta_q^{j-N} \gamma_q^{(j)} H_{pq}^{(j)} \\ \bar{H}_{p'q}^{(j)\dagger} &= \zeta^{L(N+1-j)(1-j)} \beta_q^{-j} \gamma_q^{(j)} \bar{H}_{p'q}^{(j)} \\ z(u_q)^\dagger &= (-1)^L \beta_q^2 z(t_q) \\ \alpha_q^\dagger &= \beta_q^N \alpha_q, \qquad \bar{\alpha}_q^\dagger = (-1)^L \beta_q^N \bar{\alpha}_q \\ S_q^{(j)\dagger} &= \zeta^{2r(k+1-l)} A_q^{(k,l)\dagger} X^k \hat{T}_{\bar{q}kl} \\ \tau_j(u_q)^\dagger &= \zeta^{2r(1-j)} \zeta^{L(N+1-j)(j-1)} \beta_q^{j-1} \tau_j(t_q) \end{aligned} \tag{4.8}$$

We introduce a notation similar to (2.23):

$$\begin{aligned} \langle u \rangle_{m,n} &= \prod_{j=m+1}^{n} \sin(u + j\lambda), \qquad && n \geqslant m \\ &= \prod_{j=n+1}^{m} \{\sin(u + j\lambda)\}^{-1}, \qquad && n \leqslant m \end{aligned}$$

Then

$$\begin{aligned} A_q^{(k,l)\dagger} &= \left[\frac{\eta_p^k \langle \phi_q - \theta_p \rangle_{0,l-1} \langle \theta_q - \phi_p \rangle_{-1,k-1} \langle \theta_q - \theta_{p'} \rangle_{k,N-1}}{N \eta_{p'}^l \langle \phi_q - \phi_{p'} \rangle_{-1,l-1}} \right]^L \\ H_{pq}^{(j)\dagger} &= \left\{ \frac{\eta_p^j \langle u_q - u_p \rangle_{-1,j-1}}{\sin(N\theta_p - N\phi_q)} \right\}^L \\ \bar{H}_{p'q}^{(j)\dagger} &= \left\{ \frac{\langle u_q - u_{p'} \rangle_{j-1,N-1}}{\eta_{p'}^j \sin(N\phi_q - N\phi_{p'})} \right\}^L \\ z(u_q)^\dagger &= \{\eta_p \eta_{p'} \sin(u_q - u_p) \sin(u_q - u_{p'})\}^L \\ \alpha_q^\dagger &= \{(2^{2-N}/k') \eta_q^N \sin(N\theta_p - N\phi_q) \sin(N\theta_{p'} - N\phi_q)\}^L \\ \bar{\alpha}_q^\dagger &= \{(2^{2-N}/k') \eta_q^{-N} \sin(N\theta_q - N\theta_p) \sin(N\theta_q - N\theta_{p'})\}^L \end{aligned} \tag{4.9}$$

Thus, if u_p, v_p, η_p, u_q, v_q, η_q are real, then so are these modified coefficients. Further, T_q and $S^{\dagger}_{q|kl}$ are real matrices. The property (3.14) becomes

$$z(u_q)^{\dagger}\, z(u_q+\lambda)^{\dagger}\, z(u_q+2\lambda)^{\dagger} \cdots z(u_q+N\lambda-\lambda)^{\dagger} = \alpha_q^{\dagger}\bar{\alpha}_q^{\dagger} \tag{4.10}$$

and the functional relations (3.5), (3.9), (3.11) become, for $0 \leqslant r < N-1$, $0 \leqslant j \leqslant N$,

$$T_q S_q^{(j)\dagger} = \bar{H}_{p'q}^{(j)\dagger} \tau_j(u_q)^{\dagger} + (-1)^{L+r} X^j H_{pq}^{(j)\dagger} \tau_{N-j}(u_q+j\lambda)^{\dagger} \tag{4.11}$$

$$\begin{aligned} S_q^{(j+1)\dagger} \tau_2(u_q+j\lambda)^{\dagger} = {} & [\eta_p \sin(u_n - u_p + j\lambda)]^L X S_q^{(j)\dagger} \\ & + [\eta_{p'} \sin(u_q - u_{p'} + j\lambda + \lambda)]^L S_q^{(j+2)\dagger} \end{aligned} \tag{4.12}$$

$$\begin{aligned} \tau_j(u)^{\dagger}\, \tau_2(u+j\lambda-\lambda)^{\dagger} &= Xz(u+j\lambda-\lambda)^{\dagger}\, \tau_{j-1}(u)^{\dagger} + \tau_{j+1}(u)^{\dagger} \\ \tau_j(u+\lambda)^{\dagger}\, \tau_2(u)^{\dagger} &= Xz(u+\lambda)^{\dagger}\, \tau_{j-1}(u+2\lambda)^{\dagger} + \tau_{j+1}(u)^{\dagger} \\ \tau_{N+1}(u_q)^{\dagger} &= (-1)^L\, Xz(u_q)^{\dagger} \tau_{N-1}(u_q+\lambda)^{\dagger} + (-1)^r \alpha_q^{\dagger} + (-1)^{L+r} \bar{\alpha}^{\dagger} \end{aligned} \tag{4.13}$$

from which we deduce that the $\tau_j(u)^{\dagger}$ are also real matrices. In particular, $\tau_0(u)^{\dagger} = 0$, $\tau_1(u)^{\dagger} = 1$. If the eigenvectors are also real, the corresponding eigenvalue of X being one (as must happen at least for the maximum eigenvalue of T_q for each r), then these become a set of real functional relations for the eigenvalues.

From (2.8) and (4.4), we can regard T_q as a single-valued function $T(\theta_q, \phi_q, \eta_q)$ of the related variables θ_q, ϕ_q, η_q. Then Eq. (3.15) becomes

$$\begin{aligned} \tau_2(u)^{\dagger}\, T(\theta_q+\lambda, \phi_q, \eta_q) & \\ = \left[\frac{\eta_p \eta_{p'} \sin(\theta_q - \phi_p) \sin(u_q - u_{p'})}{\sin(\theta_q - \theta_{p'})}\right]^L & X T(\theta_q, \phi_q, \eta_q) \\ + \left[\frac{\sin(\theta_q - \theta_{p'} + \lambda) \sin(u_q - u_p + \lambda)}{\sin(\theta_q - \phi_p + \lambda)}\right]^L & T(\theta_q + 2\lambda, \phi_q, \eta_q) \end{aligned} \tag{4.14}$$

The partition function Z is given by (2.11), which can be written as

$$Z = \sum (T_q \hat{T}_q)^M \tag{4.15}$$

where now the matrices T_q and $\hat{T}_q$ have been replaced by their eigenvalues, and the sum is over all eigenvalues, i.e., over all solutions of (2.13). A useful identity that follows from Eq. 2.46 of ref. 16 is

$$k' \sin N(u_q - u_p) = 2\eta_q^N \sin N(\theta_p - \phi_q) \sin N(\phi_q - \phi_p) \tag{4.16}$$

$$= 2\eta_q^{-N} \sin N(\theta_q - \theta_p) \sin N(\theta_q - \phi_p) \tag{4.17}$$

5. ZERO-TEMPERATURE LIMIT: THE LARGEST EIGENVALUES

Consider the case when k' is small, $u_p, u_{p'}$ are of order k', and $0 < u_q < \pi/N$. Then $v_p, v_{p'}$ are positive, of order k', and $v_q \simeq u_q$. If we set $k' = \varepsilon^N$, then $\mu_p, \mu_{p'}$ are of order unity, μ_q is of order ε^{-1}, and to leading order x_p, y_p, $x_{p'}$, $y_{p'}$, $x_q = 1$, $y_q = t_q$. Hence $\theta_p = \phi_p = \theta_{p'} = \phi_{p'} = \phi_q = 0$, $\theta_q = u_q = v_q$, and from (4.4), using $\bar{W}_{pq}(N) = 1$, we can deduce that, for $0 \leqslant n < N$,

$$\begin{aligned} W_{pq}(n) &= \left(\frac{\eta_p}{\eta_q}\right)^n \prod_{j=1}^{n} \frac{\sin(j\lambda - u_q)}{\sin(j\lambda)} = O(\varepsilon^n) \\ \bar{W}_{pq}(-n) &= (\eta_p \eta_q)^{-n} \prod_{j=1}^{n} \frac{\sin(u_q + j\lambda - \lambda)}{\sin(j\lambda)} = O(\varepsilon^n) \end{aligned} \tag{5.1}$$

From now on, in this section and in the Appendix we write u_q simply as u.

From (2.8) and Fig. 1, the matrix T has elements of order ε^r, occurring when the sequence $\sigma_1, \sigma'_1, \sigma_2, ..., \sigma'_L, \sigma_{L+1}$ is monotonic nonincreasing. All other elements are of higher order. Similarly for $\hat{T}$ (with the σ_J replaced by σ''_J). Hence to leading order in ε we can truncate the matrices, retaining only these elements and the corresponding configurations of the spins within a row.

For $r = 0$ (i.e., for nonskewed boundary conditions) this leaves only the states where all the spins in a row are equal to some common value σ. There are N such values, so the matrices are N by N, but they are diagonal, as the spins in adjacent rows must also be equal. In fact, since $W_{pq}(0) = \bar{W}_{pq}(0) = 1$, they are the unit N by N matrices, so all their N eigenvalues are

$$T_q = \hat{T}_q = 1 \tag{5.2}$$

For $r = 1$ the generic row spin configuration occurs when

$$\begin{aligned} \sigma_1 = \sigma_2 = \cdots = \sigma_J = \sigma \\ \sigma_{J+1} = \sigma_{J+2} = \cdots = \sigma_{L+1} = \sigma - 1 \end{aligned} \tag{5.3}$$

where $\sigma = 0, ..., N-1$ and $J = 1, ..., L$. Let $x_{\sigma|J}$, $y_{\sigma|J}$ be the corresponding elements of the eigenvector matrices $\mathbf{x}$, $\mathbf{y}$. Then the eigenvalue equations (2.13) become

$$\begin{aligned} T_q x_{\sigma|J} &= W_{pq}(1)\, y_{\sigma|J-1} + \bar{W}_{p'q}(-1)\, y_{\sigma|J} \\ \hat{T}_q\, y_{\sigma|J} &= W_{p'q}(1) x_{\sigma|J} + \bar{W}_{pq}(-1) x_{\sigma|J+1} \end{aligned} \tag{5.4}$$

where $J=1,\ldots,L$; T_q and $\hat{T}_q$ are now the eigenvalues of the transfer matrices; and we have the boundary conditions $x_{\sigma|L+1}=x_{\sigma+1|1}$, $y_{\sigma|0}=y_{\sigma-1|L}$.

These eigenvectors must also be eigenvectors of the spin-shift operator X. Let the corresponding eigenvalue of X be

$$X=\omega^{Q}, \qquad Q=0,\ldots,N-1 \tag{5.5}$$

Then for all J, $x_{\sigma|J}=\omega^{Q}x_{\sigma+1|J}$, $y_{\sigma|J}=\omega^{Q}y_{\sigma+1|J}=\omega^{Q}y_{\sigma+1|J}$. Hence the boundary conditions reduce to $x_{\sigma|L+1}=\omega^{-Q}x_{\sigma|1}$, $y_{\sigma|0}=\omega^{Q}y_{\sigma|L}$.

Equations (5.4) are simply difference equations with constant coefficients and quasiperiodic boundary conditions. Their solution is

$$x_{\sigma,J}=y_{\sigma,J}=\omega^{-Q\sigma}e^{ikJ} \tag{5.6}$$

where the wave number k is given (modulo 2π) by

$$e^{iLk}=\omega^{-Q} \tag{5.7}$$

The corresponding transfer matrix eigenvalues are

$$\begin{aligned} T_q &= e^{-ik}W_{pq}(1)+\bar{W}_{p'q}(-1) \\ \hat{T}_q &= W_{p'q}(1)+e^{ik}\bar{W}_{pq}(-1) \end{aligned} \tag{5.8}$$

Using (5.1) and defining

$$\beta=\eta_p\eta_{p'} \tag{5.9}$$

we find that these results become

$$\begin{aligned} T_q &= \frac{\beta e^{-ik}\sin(\lambda-u)+\sin u}{\eta_{p'}\eta_q\sin\lambda} \\ \hat{T}_q &= \frac{\beta\sin(\lambda-u)+e^{ik}\sin u}{\eta_p\eta_q\sin\lambda} \end{aligned} \tag{5.10}$$

There are N distinct eigenvalues of X, corresponding to $Q=0, 1,\ldots, N-1$. For each of these there are L distinct solutions of (5.7) for the wave number k (modulo 2π). Hence we have NL eigenvalues of $T_q\hat{T}_q$, as expected.

The next case is when $r=2$. Then a typical configuration of spins in a row is $\sigma, \sigma,\ldots, \sigma, \sigma-1,\ldots, \sigma-1, \sigma-2,\ldots, \sigma-2$. This can be specified by the locations J, J' of the decreases in spin value, where $1<J\leqslant J'<L$, the case $J=J'$ corresponding to a direct decrease from σ to $\sigma-2$. Hence the truncated transfer matrices are of dimension $NL(L-1)/2$, and there must

be this number of eigenvalues. Let $x_{\sigma|JJ'}$, $y_{\sigma|JJ'}$ be the corresponding elements of the eigenvectors **x**, **y**. Then we find that the eigenvectors are given by

$$x_{\sigma|JJ'} = y_{\sigma|JJ'} = \omega^{-Q\sigma}\{s(k_1, k_2)\exp(ik_1 J + ik_2 J') - s(k_2, k_1)\exp(ik_2 J + ik_1 J')\} \tag{5.11}$$

where

$$s(k, k') = e^{i(k+k')} - 2\beta\cos\lambda e^{ik'} + \beta^2 \tag{5.12}$$

and k_1, k_2 are parameters that can be regarded as defined by the boundary conditions $x_{\sigma|J,L+1} = x_{\sigma+1|1,J}$, $y_{\sigma|0,J} = y_{\sigma-1|J,L}$. These give

$$\begin{aligned} e^{iLk_1} &= -\omega^{-Q}s(k_1, k_2)/s(k_2, k_1) \\ e^{iLk_2} &= -\omega^{-Q}s(k_2, k_1)/s(k_1, k_2) \end{aligned} \tag{5.13}$$

Note that these expressions are independent of u: the eigenvectors **x**, **y** are independent of the horizontal rapidity q, as expected. We are free to multiply **x**, **y** by arbitrary constant coefficients: this affects T_q, $\hat{T}_q$, but not their product (which is the full two-row transfer matrix). The corresponding eigenvalues T_q, $\hat{T}_q$ are products of the respective right-hand sides of (5.10) over $k = k_1$ and $k = k_2$.

The expression (5.11) is a typical Bethe ansatz of the type used by Yang and Yang[30] for the anisotropic Heisenberg chain, and by Lieb[31] and Sutherland[32] for the six-vertex model. Although we have not fully verified it, the generalization to arbitrary r is presumably

$$x_{\sigma|J_1,\dots,J_r} = y_{\sigma|J_1,\dots,J_r} = \omega^{-Q\sigma}\sum_P \varepsilon_P A(k_1,\dots, k_r)e^{i(k_1J_1 + \cdots + k_rJ_r)} \tag{5.14}$$

where the sum is over all permutations P of $k_1,\dots, k_r$, ε_P being the sign (± 1) of P, and

$$A(k_1,\dots,k_r) = \prod_{1\leqslant i<j\leqslant r} s(k_i, k_j) \tag{5.15}$$

Then $k_1,\dots, k_r$ are given by the r equations

$$e^{iLk_j} = (-1)^{r-1}\omega^{-Q}\prod_{l=1}^{r} s(k_j, k_l)/s(k_l, k_j) \tag{5.16}$$

where $j = 1,\dots, r$. Taking the product of (5.16) over $j = 1,\dots, r$, we obtain

$$e^{iL(k_1 + \cdots + k_r)} = \omega^{-Qr} \tag{5.17}$$

The eigenvalues T_q, $\hat{T}_q$ are products of (5.10) over $k = k_1, \ldots, k_r$, so

$$T_q \hat{T}_q = \prod_{j=1}^{r} \frac{[\beta \sin(\lambda - u) + e^{ik_j} \sin u]^2}{\beta \eta_q^2 e^{ik_j} \sin^2 \lambda} \tag{5.18}$$

In fact, it is quite easy to verify these last results by using the same functional relaton ideas that led to the solution of the eight-vertex model.[33,25] If the sequence $\sigma_1, \sigma'_1, \sigma_2, \ldots, \sigma'_L, \sigma_{L+1}$ is monotonic nonincreasing, with $\sigma_{L+1} = \sigma_1 - r$ and $0 \leqslant r < N$, then from (2.4), (2.5), and (2.8),

$$T_q = \mu_q^{-r} F(t_q) \tag{5.19}$$

which the elements of $F(t_q)$ are polynomials in t_q of degree r. Thus in the zero-temperature limit $\varepsilon \to 0$ the relation (3.15) simplifies to

$$\tau_2(t_q)\, F(\omega t_q) = \omega^r X[\omega \mu_p \mu_{p'}(1 - t_q)]^L F(t_q) + (1 - \omega t_q)^L F(\omega^2 t_q) \tag{5.20}$$

The eigenvectors $\mathbf{x}$, $\mathbf{y}$ are independent of q, so any eigenvalue $F(t_q)$ is a linear combination of elements, and is therefore also a polynomial in t_q of degree r. Similarly, we know from ref. 16 that $\tau_2(t)$ is a polynomial of degree L, so the same is true of its eigenvalues. Regarding (5.20) as an eigenvalue relation, it follows that the RHS must vanish whenever t_q is a zero of $F(\omega t_q)$. This gives r equations which define the zeros of the polynomial $F(t_q)$. They turn out to be

$$t_q = (\omega^{1/2} \beta - e^{ik_j}) / (\omega^{-1/2} \beta - e^{ik_j}), \qquad j = 1, \ldots, r \tag{5.21}$$

where $k_1, \ldots, k_r$ are given by (5.16).

In this zero-temperature limit the commutation relation (2.12) simplifies (because f_{pq} becomes a product of a function of p and a function of q) to $T_q \hat{T}_r = T_r \hat{T}_q$. This implies that corresponding eigenvalues T_q, $\hat{T}_q$ are proportional to one another, differing at most by a constant factor. Their normalizations can be obtained by considering the cases $u = 0$ and $u = \lambda$, when the transfer matrices become proportional to the identity or the translation shift operator. Finally we obtain the result (5.18).

We have remarked that the equation are similar to those of Lieb and Sutherland. In fact (5.16) is precisely the equation for a six-vertex model in a horizontal field [Eqs. (136)–(147) of ref. 34; Eqs. (7.69) and (1.26) of ref. 35], except that $s(k, k')$ is replaced by $s(k', k)$, or equivalently L by $-L$. The field disappears when $\beta = 1$. We know[15] that the N-state chiral Potts model is related to such a six-vertex model. Here is another manifestation of this connection.

We shall find it convenient to make the standard "transformation to a difference kernel,"(36,30,31) transforming from k_j to α_j, where

$$e^{ik_j} = \beta \sin(\alpha_j - \lambda)/\sin \alpha_j \tag{5.22}$$

for $j = 1,..., r$. Then $F(t_q)$ has zeros when (5.21) is satisfied, i.e., when $t_q = e^{2i\alpha_j}$, and

$$\frac{s(k_j, k_l)}{s(k_l, k_j)} = \frac{\sin(\lambda - \alpha_l + \alpha_j)}{\sin(\lambda - \alpha_j + \alpha_l)} \tag{5.23}$$

and (5.16), (5.18) become

$$\left(\frac{\sin(\alpha_j - \lambda)}{\sin \alpha_j}\right)^L = (-1)^{r-1} \beta^{-L} \omega^{-Q} \prod_{l=1}^{r} \frac{\sin(\alpha_j - \alpha_l + \lambda)}{\sin(\alpha_l - \alpha_j + \lambda)} \tag{5.24}$$

$$T_q \hat{T}_q = \prod_{j=1}^{r} \frac{\sin^2(u - \alpha_j)}{\eta_q^2 \sin \alpha_j \sin(\alpha_j - \lambda)} \tag{5.25}$$

5.1. Interfacial Tension

Now consider the partition function (2.11) and indicate its dependence on the skew parameter r by writing it as Z_r. For $k < 1$ and $r = 0$ we expect the system to be ferromagnetically ordered into one of N possible phases, all of equal free energy, phase σ having a preponderance of spins with value σ. (More precisely, in phase σ the probability of any spin having the particular value σ is greater than $1/N$.)

For $r = 1$ we still the expect the system to be ordered, but now the boundary condition (2.10) is inconsistent with the whole system being in a single phase. Rather, there must be an interface, running approximately vertically through the lattice, separating a σ phase on the left from a $\sigma - 1$ phase on the right.

For $r = 2$ there may be a single interface separating a σ phase from a $\sigma - 2$ phase, or alternatively two interfaces, separating σ, $\sigma - 1$, and $\sigma - 2$ phases, as in Fig. 2.

For $j = 1,..., N - 1$, let e_j be the interfacial tension (i.e., the surface energy) of a surface of type j, i.e., a boundary between phases σ and $\sigma - j$. (Because of Z_N invariance, e_j is independent of σ.) Take L (the number of sites per row) and M (the number of rows) to be large, with $L \gg M$. Then we expect Z_r/Z_0 to be a polynomial in L of degree r, expandable as

$$Z_r/Z_0 = \sum c(n_1,..., n_{N-1}) \exp\left\{-\sum n_j e_j M/k_B \mathscr{T}\right\} \tag{5.26}$$

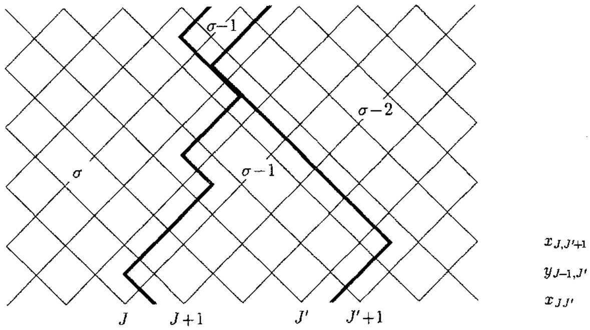

Fig. 2. The r interfaces forced on the model by the skewed boundary conditions (for $r=2$). In the zero-temperature limit considered here they must move monotonically up the lattice, so each interface passes through each row just once. They may coalesce.

where n_j is the number of interfaces of type j, k_B is Boltzmann's constant, and $\mathscr{T}$ is the temperature. These n_j must satisfy $\sum jn_j = r$. For L large the coefficient $c(n_1,..., n_{N-1})$ is the number of ways of locating the various interfaces on the columns $1,..., L$, so is a polynomial in L of degree $n_1 + \cdots + n_{N-1}$. In particular, for L and M large

$$\begin{aligned} Z_1/Z_0 &\sim d_0\, e^{-\varepsilon_1 M/k_B \mathscr{T}} \\ Z_2/Z_0 &\sim (d_1^2/2)\, e^{-2\varepsilon_1 M/k_B \mathscr{T}} + d_2 e^{-\varepsilon_2 M/k_B \mathscr{T}} \end{aligned} \tag{5.27}$$

where $d_0 \simeq d_1$ and (to within factors that decay as powers of M) $d_0, d_1, d_2 \sim L$.

The coefficients of highest degree in L is $c(r, 0,..., 0) \sim L^r/r!$. Letting L become large, it follows that

$$Z_r/Z_0 \sim (L^r/r!)\, e^{-rMe_1/k_B \mathscr{T}} \tag{5.28}$$

and hence

$$e_1/k_B \mathscr{T} = -\lim_{M\to\infty} M^{-1} \lim_{L\to\infty} \ln(r!\, L^{-r} Z_r/Z_0) \tag{5.29}$$

Note that the limit $L \to \infty$ is to be taken *before* the limit $M \to \infty$.

We expect these observations to be correct for all subcritical temperatures $\mathscr{T}$, i.e., for $k' < 1$. For $\mathscr{T} = k' = 0$ we can use our previous results to calculate the ε_j. The easiest one to obtain is ε_1, since this can be calculated from just Z_0 and Z_1. For $r=0$, the transfer matrices are the N by N unit matrix, so from (2.11) (in the zero-temperature limit $k' \to 0$)

$$Z_0 = N \tag{5.30}$$

For $r=1$, write e^{ik_1} as z. Then from (4.15) and (5.18)

$$Z_1 = \sum \left[\beta \sin(\lambda - u) + z \sin u\right]^{2M} / (\beta \eta_q^2 z \sin^2 \lambda)^M \tag{5.31}$$

where the sum is over all the eigenvalues of $T_q \hat{T}_q$, i.e., over all solutions for z of (5.7), i.e., of $z^L = \omega^{-Q}$. For a given value of Q this equation has L distinct solutions, corresponding to all the eigenvalues in the sector with $X = \omega^Q$. Letting Q take all its allowed values $0, \ldots, N-1$, we see that the sum in (5.31) is over all the NL solutions of

$$z^{NL} = 1 \tag{5.32}$$

It follows at once that for $-NL < j < NL$

$$\sum z^j = NL\delta_{j,0} \tag{5.33}$$

then sum also being over all solutions of (5.32).

Using the binomial theorem to expand the numerator in (5.31) and defining

$$\xi = \sin u \sin(\lambda - u)/(\eta_q \sin \lambda)^2 \tag{5.34}$$

provided $M < NL$, we obtain the exact result

$$Z_1 = NL \binom{2M}{M} \xi^M \tag{5.35}$$

Substituting into (5.29), taking $r=1$, it follows that

$$\varepsilon_1 / k_B \mathscr{T} = -\ln(4\xi) \tag{5.36}$$

For $r > 1$, we can obtain a simple upper bound for ε_r by noting that if we only consider row states where the spins jump immediately from σ to $\sigma - r$ (with no $\sigma - 1, \ldots, \sigma - r + 1$ spins in between), then we obtain Eqs. (5.4)–(5.8), but with $W_{pq}(1)$, $\bar{W}_{pq}(-1)$ replaced by $W_{pq}(r)$, $\bar{W}_{pq}(-r)$. The Z_r calculated in this way is a restricted sum over configurations; since all the Boltzmann weights are positive (provided $0 < u_q < \lambda$), it is therefore less than the true partition function Z_r. Noting that in the zero-temperature limit $W_{pq}(r)\, \bar{W}_{pq}(-r) = W_{p'q}(r)\, \bar{W}_{p'q}(-r)$, it follows that

$$\varepsilon_r / k_B \mathscr{T} < -\ln[4 W_{pq}(r)\, \bar{W}_{pq}(-r)] \tag{5.37}$$

for $r = 2, \ldots, N-1$. In particular, this implies

$$\frac{\varepsilon_2 - 2\varepsilon_1}{k_B \mathscr{T}} < -\ln\left(\frac{\sin^2 \lambda \sin(u + \lambda) \sin(2\lambda - u)}{4 \sin^2(2\lambda) \sin u \sin(\lambda - u)}\right) \tag{5.38}$$

The RHS is negative for u sufficiently close to 0 or λ, so at least for these values $\varepsilon_2 < 2\varepsilon_1$. This means that the system prefers to go straight from phase 2 to phase 0, rather than from 2 to 1 to 0: phase 1 does not wet the interface between phases 2 and 0.

The partition function Z_2 is calculated exactly (in the zero-temperature limit) in the Appendix, being given by (A10), (A12): we finid that it does indeed have the form (5.27), and $\varepsilon_2 < 2\varepsilon_1$ for *all* u between 0 and λ, so phase 1 does not wet the (0, 2) interface.

Further investigation of the $r=2$ case shows that for L large there are two types of eigenvector: "plane waves," in which k_1 and k_2 are both real, and "bound states," in which they are complex and either $s(k_1, k_2)$ or $s(k_2, k_1)$ vanishes. The number of the first (second) type is of order L^2 (L). The two types give the two contributions to Z_2 in (5.27).

This suggests that for arbitrary r (from 2 to $N-1$) the ε_r contribution to Z_r comes from states in which all the r dislocations are bound together. This will happen if $s(k_{j+1}, k_j)=0$ for $j=1,\ldots, r-1$ and $\mathrm{Im}(k_r - k_1) > 0$. Equations (5.16) are then individually satisfied and (5.14) reduces to a single sum, only the identity permutation giving a nonzero contribution.

From (5.23) we must have $\alpha_{j+1} = \alpha_j - \lambda$, so

$$\alpha_j = (\pi/2) - \chi - I_j, \qquad j=1,\ldots, r \tag{5.39}$$

where

$$I_j = (j-1-r/2)\lambda \tag{5.40}$$

and the parameter χ is determined by (5.17), i.e.,

$$\beta^r \frac{\cos(\chi + r\lambda/2)}{\cos(\chi - r\lambda/2)} = \rho \tag{5.41}$$

where

$$\rho^L = \omega^{-Qr} \tag{5.42}$$

When $\eta_p = \eta_{p'} = 1$ and $Q=0$, then $\beta=1$ and this equation has the simple solution $\chi=0$. In this case

$$e^{ik_j} = \frac{\cos(j-r/2)\lambda}{\cos(j-1-r/2)\lambda}, \qquad j=1,\ldots, r \tag{5.43}$$

Remembering the definition (4.1), $\lambda = \pi/N$, and that $r<N$, we have that the RHS of (5.43) is always positive real, so we can take $k_1,\ldots, k_r$ to be pure imaginary. They are distributed symmetrically about zero: $k_{r+1-j} = -k_j$ and $\mathrm{Im}(k_r) > 0$.

To within an irrelevant normalization factor independent of σ, $J_1,..., J_r$, the RHS of (5.14) is therefore always positive, so the eigenvectors **x**, **y** have positive entries. The transfer matrices T_q, $\hat{T}_q$ also have positive entries (for $0<u<\lambda$), so from the Perron–Frobenius theorem we have found the largest eigenvalue of $T_q\hat{T}_q$ in the large-L limit.

Further, if $u=\lambda/2$, then $\hat{T}_q=T_q^\dagger$, so $T_q\hat{T}_q$ is real and symmenric. All its eigenvalues are therefore real, so for M (and L) large the sum in (4.15) is dominated by the contribution from the largest eigenvalue. Using also (5.26), (5.25), and (5.39), it follows that

$$\begin{aligned}\frac{\varepsilon_r}{k_B\mathscr{T}} &= -\ln T_q\hat{T}_q \\ &= -\sum_{j=1}^{r} \ln\left(\frac{\cos^2(I_j+I_{j+1})/2}{\eta_q^2\cos I_j \cos I_{j+1}}\right) \end{aligned} \tag{5.44}$$

The contribution to Z_r from ε_r is (for large M) greater that that from all the other terms in (5.26), so there is no wetting by intermediate phases.

From (5.39) the α_j can be regarded as forming a horizontal "string" of length r in the complex plane. Such solutions of the Bethe ansatz are well known, but usually they correspond to excited states of the system: here they correspond to the ground state (i.e., the largest eigenvalue of the transfer matrix). This difference can be ascribed to the fact that our $s(k, k')$ is that of the six-vertex model, *but with k and k' interchanged.*

The situation is mathematically more complicated for $u\neq\lambda/2$, $0<u<\lambda$, although we do not expect any qualitative change in the physical behavior. The eigenvalues are then no longer real, so when both M and L become large it is possible for the contribution to (4.15) of the largest eigenvalue to be canceled by those from nearby eigenvalues with different phases. This happens in the calculation of the correlation length of the eight-vertex and hard-hexagon models[37,24] and here we use similar techniques to handle the problem.

We have to keep all the allowed bound-state eigenvalues given by (5.39) and (5.41). First continue to suppose that $|\beta|=1$. Then $\mathrm{Re}(\chi)$ is either 0 or $\pi/2$. The latter case violates the restriction $\mathrm{Im}(k_r-k_1)>0$, while the former satisfies it. Hence χ lies on the imaginary axis.

As $L\to\infty$, the solutions of (5.41) for χ become infinitely dense everywhere on the imaginary axis, so the sum (4.15) becomes an integral over χ along he imaginary axis (weighted by some M-independent density function).

Regard $T_q\hat{T}_q$, as given by (5.25) and (5.39), as a function of χ. It appears that it has a ridge along the real axis of the complex χ plane, with a saddle point χ_0 between $(r\lambda-\pi)/2$ and $(\pi-r\lambda)/2$. (For $u=\lambda/2$ it is at the origin.) At the saddle point the derivative vanishes, i.e.,

$$\sum_{j=1}^{r} \{2\tan(\chi_0+u+I_j)-\tan(\chi_0+I_j)-\tan(\chi_0+I_{j+1})\}=0, \qquad |\chi_0|<(\pi-r\lambda)/2 \tag{5.45}$$

which conditions can be regarded as defining the real parameter χ_0. It should be a continuous function of u, zero when $u=\lambda/2$.

Hence the χ integration can be deformed so as to cross the ridge at the saddle, doing so along a part wherte $T_q\hat{T}_q$ is real and positive. In the limit of M large, the integral is then dominated by the contribution from the saddle point (where $T_q\hat{T}_q$ is maximized on the path), so

$$\begin{aligned} &Z_r \sim (T_q\hat{T}_q)^M|_{\chi=\chi_0}, \\ &\frac{\varepsilon_r}{k_B\mathscr{T}} = -\sum_{j=1}^{r} \ln\left(\frac{\cos^2(\chi_0+u+I_j)}{\eta_q^2\cos(\chi_0+I_j)\cos(\chi_0+I_{j+1})}\right) \end{aligned} \tag{5.46}$$

If $|\beta|\neq 1$, then the contour of integration shifts off the imaginary axis, but the saddle point is unaltered (because $T_q\hat{T}_q$ is independent of β). So long as the contour does not cross any singularities in the χ plane, there should therefore be no change in the large-M behavior of the integral: ε_r should be independent of β (as indeed we have observed for $r=1$ and $r=2$).

For $r=1$ and 2 we have verified that (5.45) and (5.46) agree with our explicit results for Z_1 and Z_2: more detail is given at the end of the Appendix.

Another interesting case is when $|\eta_p|=|\eta_{p'}|=1$ and $u-\lambda/2$ is pure imaginary. Then $\hat{T}_q=T_q^\dagger$, so the matrix $T_q\hat{T}_q$ is Hermitian. The eigenvalues $T_q\hat{T}_q$ are therefore real and again the sum in (4.15) is dominated by the largest eigenvalue. In this case the integration is still along the imaginary χ axis, but now the saddle point lies on this axis, so there is no need to deform the contour. One still has to locate it by solving (5.45) (modifying the subsidiary condition on its location), and again ε_r is then given by (5.46).

Using (5.9), note that the vertical rapidities p, p' enter the transfer matrix elements (in this zero-temperature limit) only via β. Hence independence of β is here equivalent to independence of p and p'. In fact we expect this to be true from "Z-invariance" arguments[38] in which we allow the rapidities of all the vertical lines to be different. Provided the system is "physical" (certainly if all the Boltzmann weights are positive), the free energy of a vertical interface should only depend on the vertical rapidities near it. However, because the column-to-column transfer matrices commute, the partition function is unchanged by permuting the vertical rapidities, so any rapidity can be moved to a line far from the interface. It follows that each ε_r must be independent of all the vertical rapidities, and hence of p and p'.

This independence may be useful in calculating the interfacial tensions for nonzero temperature. It appears that we should be able to obtain them in general by considering only the "superintegrable case," when $x_{p'} = y_p$, $y_{p'} = x_p$, and $\mu_{p'} = 1/\mu_p$.(2-14) In this case $\theta_{p'} = \phi_p$, $\phi_{p'} = \theta_p$, $\eta_p \eta_{p'} = e^{i\lambda}$, so η_p and $\eta_{p'}$ cannot both be real and we are ouside the physical domain in which the Boltzmann weights are real and positive. We have to be careful about using Z-invariance arguments. However, the zero-temperature solution considered here can certainly be analytically continued to arbitrary complex values of η_p and $\eta_{p'}$ (of order unity), so contains a superintegrable case. This suggests that Z invariance can indeed be used, and the interfacial tensions obtained for the general model from the superintegrable case (at least within appropriate domains in the complex p, p', q planes).

6. SUPERINTEGRABLE CASE

Setting $k = l = j = 0$ in (3.1)–(3.5) and (2.26), we see that $T_q \hat{T}_{\bar{q}00}$ is proportional to the matrix $\tau_N(t_q)$. This is a simpler matrix function than T_q itself, being just a polynomial in t_q, determinable from (3.11). [In refs 17 and 18 it is pointed out that the functional relations can be solved by first solving (3.11) for the τ matrices, then solving (3.5) for T_q.] The model therefore simplifies if the row rapidities alternate, being successively $q = \{x_q, y_q, \mu_q\}$ and $\bar{q}00 = \{y_q, x_q, \mu_q^{-1}\}$. If we rotate the lattice through 90°, then it follows that simplifications should occur (for *all* row rapidities q) when the alternating column rapidities p and p' satisfy

$$x_{p'} = y_p, \qquad y_{p'} = x_p, \qquad \mu_{p'} = 1/\mu_p \tag{6.1}$$

Indeed they do, This is the "superintegrable" case. It (more specifically the homogeneous superintegrable case, when $x_p = y_p$, $\mu_p = 1$) has been discussed in a sequence of papers.(2-14) In Eq. (2.22) of ref. 5 and Eq. (2.21) of ref. 14, McCoy *et al.* have proposed an ansatz for the from of the eigenvalue function T_q. Here we show that the functional relations simplify for the superintegrable case, and that they imply this ansatz.

The general method for solving the functional relations is given in refs. 17 and 18; here we specialize it to the superintegrable case. First we consider the algebraic form of the eigenvalue function T_q. From (2.4), (2.5), and (2.8), all elements of the transfer matrix T depend on q as $\mu_q^{-r} \times$

(rational function of x_q, μ_q). Since the eigenvectors are independent of q, each eigenvalue T_q must also have this form. The only poles are when $x_q^N = y_p^N$, or equivalenbtly $y_q^N = x_p^N$. These poles are counted in ref. 18, except that here we should replace p in $\bar{W}, \bar{\Theta}$ therein by p' (which makes $\bar{\Theta}_{ij} = \Theta_{ij}$). We find that the poles of T_q are the zeros of $(x_q^N - y_p^N)^L/(x_q - y_p)^L$. Similar considerations (with p replaced by p') apply for $\hat{T}_q$. It follows that

$$\begin{aligned} T_q &= N^{L/2}\mu_q^{-r}\frac{(x_q - y_p)^L}{(x_q^N - y_p^N)^L}\mathscr{T}(x_q, y_q) \\ \hat{T}_q &= N^{L/2}\mu_q^{-r}\frac{(x_q - x_p)^L}{(x_q^N - x_p^N)^L}\hat{\mathscr{T}}(x_q, y_q) \end{aligned} \tag{6.2}$$

where $\mathscr{T}(x_q, y_q)$, $\hat{\mathscr{T}}(x_q, y_q)$ are polynomials (multinomials) in both x_q and y_q, e.g., $\mathscr{T}(x, y) = \sum_i \sum_j c_{ij} x^i y^j$, where the sums are over a finite number of nonnegative values of i and j. We can use the relation (2.2) to ensure that i and j do not both exceed $N-1$.

[We have used the fact that if $P(x, y)$ is a polynomial in variables x and y satisfying (2.2), i.e., $x^N + y^N = k(1 + x^N y^N)$, and if $P(x, y)$ vanishes when $x = x_0$ for all the N corresponding values of y, then $P(x, y)/(x - x_0)$ is also a polynomial in x and y.]

Using Eqs. (2.48) and (2.49) of ref. 16, we can establish that

$$\frac{f_{p'q}}{f_{pq}} = \mu_p^{N-1}\frac{x_q - y_p}{x_q - x_p}\frac{x_q^N - x_p^N}{x_q^N - y_p^N} \tag{6.3}$$

The commutation relation (2.12) therefore simplifies to

$$\mathscr{T}(x_q, y_q)\,\hat{\mathscr{T}}(x_r, y_r) = \mathscr{T}(x_r, y_r)\,\hat{\mathscr{T}}(x_q, y_q), \qquad \forall q, r \tag{6.4}$$

which implies that $\hat{\mathscr{T}}(x_q, y_q)$ is proportional to $\mathscr{T}(x_q, y_q)$.

Substituting into (2.22), (2.26), we obtain

$$\mathscr{T}(\omega x_q, \omega^{-1} y_q) = \omega^{-Q-r-L}\mathscr{T}(x_q, y_q) \tag{6.5}$$

$$S_q^{(j)} = N^{-L/2}\mu_q^r \mu_p^{jL}(y_p - x_q)^{-L}\,\hat{\mathscr{T}}(y_q, \omega^j x_q) \tag{6.6}$$

and the functional relations (3.5) become, for $0 \leqslant j \leqslant N$,

$$\begin{aligned} &\mathscr{T}(x_q, y_q)\,\hat{\mathscr{T}}(y_q, \omega^j x_q) \\ &\quad = (t_p, t_q)^L_{j-1, N-1}\,\tau_j(t_q) + \omega^{j(r+Q+L)}(t_p, t_q)^L_{-1, j-1}\,\tau_{N-j}(\omega^j t_q) \end{aligned} \tag{6.7}$$

Let

$$J(x_q, y_q) = \prod_{j=0}^{N-1} \mathscr{T}(x_q, \omega^j y_q) \tag{6.8}$$

This is a polynomial in x_q and y_q. Using (6.5), we can establish that $J(x_q, y_q) = J(\omega x_q, y_q) = J(x_q, \omega y_q)$, from which it follows that $J(x_q, y_q)$ is a polynomial in x_q^N and y_q^N. Using (2.2), we can therefore write it as a Laurent polynomial in the single variable μ_q^N.

As in (2.3), define $\Lambda_q = \mu_q^N$. Consider the limiting case when $\mu_q \to 0$, which from (2.2) implies that $x_q \to \infty$, while $x_q \mu_q$ and y_q tend to finite nonzero limits. Then $W_{pq}(n)$, $\bar{W}_{p'q}(n)$, and T_q also tend to limits, so from (6.2), $\mathscr{T}(x_q, y_q)$ diverges at most as strongly as $\mu_q^{r-(N-1)L}$, and $J(x_q, y_q)$ as $\Lambda_q^{r-(N-1)L}$. Likewise, when $\mu_q \to \infty$, then $J(x_q, y_q)$ grows no faster than Λ_q^r. It follows that

$$J(x_q, y_q) = \Lambda_q^{r-(N-1)L} \mathscr{G}(\Lambda_q) \tag{6.9}$$

where $\mathscr{G}(\Lambda)$ is a polynomial of degree at most $(N-1)L$. $\mathscr{G}(0)$ may be zero.

Similarly, interchanging x_q and y_q, inverting μ_q, and remembering that $\hat{\mathscr{T}} \propto \mathscr{T}$, we find

$$\prod_{j=0}^{N-1} \hat{\mathscr{T}}(y_q, \omega^j x_q) = \Lambda_q^{-r+(N-1)L} \hat{\mathscr{G}}(\Lambda_q^{-1}) \tag{6.10}$$

where $\hat{\mathscr{G}}(\Lambda) \propto \mathscr{G}(\Lambda)$.

The RHS of (6.7) is a polynomial in $t_q = x_q y_q$ of degree at most $(N-1)L$. Taking the product of (6.7) over $j = 0, \ldots, N-1$ and using (6.10), we obtain

$$\mathscr{T}(x_q, y_q) = [\Lambda_q^{r-(N-1)L} \mathscr{F}(t_q) / \hat{\mathscr{G}}(\Lambda_q^{-1})]^{1/N} \tag{6.11}$$

where $\mathscr{F}(t_q)$ is a polynomial of degree not greater than $N(N-1)L$.

From (2.2), t_q and Λ_q are related by

$$k^2 t_q^N = 1 + k'^2 - k'(\Lambda_q + \Lambda_q^{-1}) \tag{6.12}$$

Thus, if $\mathscr{F}(t_q)$ contains a factor $t_q^N - t_0^N$, then this can cancel factors $\Lambda_q - \Lambda_0$ and/or $\Lambda_q - \Lambda_0^{-1}$ in $\hat{\mathscr{G}}(\Lambda_q^{-1})$, where t_0, Λ_0 are constants related by (6.12). Such cancellations are obviously necessary to ensure that $\mathscr{T}(x_q, y_q)$ is finite for all finite values of x_q, y_q.

Further, $\mathscr{T}(x_q, y_q)$ is single-valued. Counting the zeros and poles of $\mathscr{F}(t_q)/\hat{\mathscr{G}}(\Lambda_q^{-1})$ (one way is to use to Θ, $\bar{\Theta}$ functions introduced in ref. 18), we find that reduces to

$$\mathscr{T}(x_q, y_q) = x_q^{P_a} y_q^{P_b} \Lambda_q^{-P_c} F(t_q)\, G(\Lambda_q^{-1}) \tag{6.13}$$

where $F(t_q)$, $G(\Lambda_q)$ are polynomials of degree m_P, m_E, respectively; P_a, P_b, P_c are integers, and without loss of generality we can choose $F(0)$, $G(0)$, $G(k')$, $G(1/k')$ to be nonzero.

Substituting this form into (6.2), we obtain

$$T_q = N^{L/2} \frac{(x_q - y_p)^L}{(x_q^N - y_p^N)^L} x_q^{P_a} y_q^{P_b} \mu_q^{-P_\mu} F(t_q)\, G(\mu_q^{-N}) \tag{6.14}$$

where $P_\mu = r + NP_c$.

Again considering the limiting cases $\mu_q \to 0$ and ∞, we obtain the inequalities

$$P_b + m_P \leqslant P_\mu \leqslant (N-1)L - P_a - Nm_E - m_P \tag{6.15}$$

Also, considering the limits $x_q \to 0$ and $y_q \to 0$, we find that P_a, P_b must be nonnegative, as of course are m_P and m_E. From (6.15), so therefore is P_μ. From the definition of P_μ and from (6.5),

$$\begin{aligned} P_\mu &= r, \quad \bmod N \\ P_b - P_a &= Q + r + L, \quad \bmod N \end{aligned} \tag{6.16}$$

The result (6.14) is precisely the ansatz postulated by Albertini *et al.* in Eq. (2.22) of ref. 5 for the case $r = 0$. Then (6.15) is their inequalities (2.27), while (6.16) is consistent with their observations (2.26).

The case r, $m_P = 0$, when $F(t)$ is a constant, was discussed in ref. 3; the m, x therein are our $m_E, x_q/y_p$. In this case $P_b = P_\mu = 0$ and $P_a = (N-1)L - Q$, mod N (i.e., $0 \leqslant P_a < N$).

Functional Relations for F and G. We now substitute these forms into the functional relations (6.7) and (3.9)–(3.15). Since $\hat{\mathscr{T}}$ is proportional to $\mathscr{T}$, it can be taken as also given by (6.13), but with G replaced by $\hat{G}$, where $\hat{G}(\Lambda) \propto G(\Lambda)$. We obtain

$$\begin{aligned} &t_q^{P_a + P_b} F(t_q) F(\omega^j t_q)\, G(\Lambda_q^{-1})\, \hat{G}(\Lambda_q) \\ &\quad = \omega^{-jP_b} (t_p, t_q)^L_{j-1, N-1}\, \tau_j(t_q) + \omega^{-jP_a} (t_p, t_q)^L_{-1, j-1}\, \tau_{N-j}(\omega^j t_q) \end{aligned} \tag{6.17}$$

Both (3.9) and (3.15) simplify to

$$\tau_2(t_q)\, F(\omega t_q) = \omega^{-P_a} (t_p - t_q)^L F(t_q) + \omega^{P_b} (t_p - \omega t_q)^L\, F(\omega^2 t_q) \tag{6.18}$$

while the τ relations (3.11) are unchanged, except that the definitions (3.12), (3.13) reduce to

$$z(t_q) = \omega^L (t_p - t_q)^{2L} \tag{6.19}$$

$$\alpha_q = \bar{\alpha}_q = (t_p^N - t_q^N)^L \tag{6.20}$$

Apart from simple constants, the relation (6.18) is the same as (5.20), and is very similar to the corresponding equation for the six-vertex model—e.g., Eq. (9.4.3) of ref. 25. One can solve it in the same way. As in ref. 5, let

$$F(t_q) = \prod_{l=1}^{m_P} (1 + \omega v_l t_q / t_p) \tag{6.21}$$

where $v_1, \ldots, v_{m_P}$ are constants. Setting $t_q = -\omega^{-2} t_p / v_j$ in (6.18), we find that the LHS vanishes and we get, for $j = 1, \ldots, m_P$,

$$\left(\frac{v_j + \omega^{-1}}{v_j + \omega^{-2}}\right)^L = -\omega^{-P_a - P_b} \prod_{l=1}^{m_P} \frac{v_j - \omega^{-1} v_l}{v_j - \omega v_l} \tag{6.22}$$

[when $N = 3$ this is precisely Eq. (4.4) of ref. 5]. Making the transformation corresponding to (5.22), i.e., $v_j = -\omega^{-1} e^{-2i\alpha_j}$, and remembering that $\lambda = \pi/N$, we get

$$\left(\frac{\sin(\alpha_j - \lambda)}{\sin \alpha_j}\right)^L = -\omega^{P_a + P_b + m_P + L/2} \prod_{l=1}^{m_P} \frac{\sin(\alpha_j - \alpha_l + \lambda)}{\sin(\alpha_j - \alpha_l - \lambda)} \tag{6.23}$$

This is precisely Eq. (5.24) of Section 5, but with $\beta^{-L} \omega^{-Q}$ replaced by $\omega^{P_a + P_b + m_P + L/2}$ and r by m_P.

Equations (6.23) define $\alpha_1, \ldots, \alpha_{m_P}$. They have the remarkable property that they are temperature independent. Apart from the rather trivial phase factor outside the product, they depend on the original Boltzmann weights only via N. This is not a new observation—it is presumably related to the connection of the chiral Potts model with the six-vertex model.[15]

Further, at least for N odd, these are also the equations that occur in the critical case ($k = 0$) of the general chiral Potts model, namely the Fateev–Zamolodchikov model. This case has been studied by Albertini—Eq. (23) of ref. 13—who also observes that the largest eigenvalue of the transfer matrix corresponds to the α_j being grouped into strings of maximal length ($N - 1$ for his case; r for the zero-temperature case we considered in Section 5).

Once (6.23) has been solved for $\alpha_1, \ldots, \alpha_{m_P}$ (and hence for $v_1, \ldots, v_{m_P}$), one still needs to calculate the functions $G(\Lambda_q)$, $\hat{G}(\Lambda_q)$ (with differ only by a constant factor). To do this one uses (6.18) and (3.11) to express the functions $\tau_j(t_q)$ in terms of $F(t_q)$. Then we find that (6.17), for *all* values of j, becomes the single equation

$$G(\Lambda_q)\, \hat{G}(\Lambda_q^{-1}) = \mathscr{P} \tag{6.24}$$

where

$$\mathscr{P} = \omega^{-P_b} \sum_{k=0}^{N-1} \frac{(t_p^N - t_q^N)^L (\omega^k t_q)^{-P_a - P_b}}{(t_p - \omega^k t_q)^L\, F(\omega^k t_q)\, F(\omega^{k+1} t_q)} \tag{6.25}$$

Plainly P is a rational function of t_q. [For the case when r, $m_P = 0$ it is proportional to the function $P(z^N)$ defined in (2.11) of ref. 6, with $z = t_p/t_q$.] In fact, from (6.24) and (6.12) it must reduce to a polynomial in t_q^N of degrre m_E, so it can be written as

$$\mathscr{P} = C \prod_{k=1}^{m_E} (w_k - \Lambda_q)(w_k - \Lambda_q^{-1}) \tag{6.26}$$

where C, $w_1,..., w_{m_E}$ are some constants. Then (6.24) has the 2^{m_E} solutions

$$G(\Lambda_q),\ \hat{G}(\Lambda_q) \propto \prod_{k=1}^{m_E} (\Lambda_q - w_k^{\pm 1}) \tag{6.27}$$

where the m_E signs can be chosen independently.

These properties have been discussed by Tarasov[7]; for $N = 3$ by McCoy *et al.*[5,8,9,14]; and for the $m_P = 0$ and $N = 2$ (Ising) cases by the author.[3,11] The superintegrable chiral Potts model seems top amalgamate features of the Ising model (notably the factorization of the eigenvalues of T_q into functions of Λ_q and functions of t_q, and the independent choices of the Λ_q factors) with those of the six-vertex model (the Bethe-ansatz-type equations for the zeros of the t_q function).

The normalizations of G and $\hat{G}$ can be deduced from (6.24) (to within the usual arbitrariness of allowing the transformation $T_q \to c T_q$, $\hat{T}_q \to c^{-1} \hat{T}_q$, which leaves the full two-row transfer matrix unchanged).

The superintegrable model contains a Hermitian case, when η_q, μ_q, v_p, $v_{p'}$ are real; $u_q - \lambda/2$, $v_q - \lambda/2$, u_p, $u_{p'}$ are pure imaginary; and η_p, $\eta_{p'}$ are unimodular. This ensures that $W_{pq}(n)^* = \bar{W}_{pq}(-n)$ and $T_q^\dagger = \hat{T}_q$, so all the eigenvalues of $T_q \hat{T}_q$ must be real. Considering this case, we can verify that the w_k must then be real, for all allowed p, p'. This in turn implies (for the general superintegrable model) that the zero of $\mathscr{P}$ always lie on the negative real axis in the t_q^N/t_p^N plane.

7. CONCLUSIONS

We have extended the transfer matrix functional relations for the chiral Potts model to skewed boundary conditions. (An interesting feature is that they have the same set of solutions as before, but with different rules for determining which solutions to use.)

The zero-temperature calculation of Section 5 supports two conclusions. First, the vertical interfacial tension, as defined in (5.26), is an analytic function of the vertical rapidities p and p' in a domain that spans both the superintegrable case (6.1) and the physical case (when the Boltzmann weights are positive real). Second, within this domain it is actually independent of p and p', in agreement with Z invariance.[38]

In Section 6 we have specialized the functional relations to the superintegrable case, when simplifications occur. The next step, which we hope to perform, is to calculate the interfacial tension in this case, guided by the working of Section 5 when selecting the required maximum eigenvalues. From the Z-invariance argument, this should be independent of the remaining variable p: in fact, it should be the interfacial tension of the general physical chiral Potts model.

APPENDIX

A1. Calculation of Z_2 and ϵ_4

To calculate Z_2 in the zero-temperature limit, consider the Bethe ansatz equations (5.16), taking $r=2$. If we set $z_m = e^{ik_m}$, $\rho = z_1 z_2$, then for $j=1$, (5.16) gives

$$2\beta \cos \lambda (z_1^{L+2} + \omega^{-Q}\rho) - (\rho + \beta^2)(z_1^{L+1} + \omega^{-Q} z_1) = 0 \tag{A1}$$

Also, taking the product of (5.6) over $j=1$ and 2, we get

$$\rho^L = \omega^{-2Q} \tag{A2}$$

Suppose m and ρ are given. Then (A1) has $L+2$ solutions for z_1. Summing over these solutions, we obtain

$$\begin{aligned} \sum z_1^j &= \left(\frac{\rho + \beta^2}{2\beta \cos \lambda}\right)^j, && 0 < j \leqslant L \\ &= L+2, && j=0 \\ &= \left(\frac{\rho + \beta^2}{2\rho\beta \cos \lambda}\right)^{-j}, && -L \leqslant j < 0 \end{aligned} \tag{A3}$$

Also, if we sum ρ^k over all solutions of (A2), for all values $0,..., N-1$ of Q, we obtain

$$\sum \rho^k = NL\delta_{k,0} \tag{A4}$$

provided only that $-NL < k < NL$.

Replacing j in (A3) by $i-j$, multiplying by ρ^j, summing over the values of ρ, and using (A4), at first sight we obtain the sum of $z_1^i z_2^j$ over the solutions of (5.16). However, this spuriously includes solutions with $z_1 = z_2$. There are NL of these, being the solutions of

$$z_1^L = -\omega^{-Q}, \qquad Q = 0,..., N-1 \tag{A5}$$

and for $|i|$, $|j| < NL$ these give an unwanted contribution $NL\delta_{i,-j}$. Subtracting this off, the remainder overcounts the true sum by a factor of two, since for every solution for z_1 and z_2 it includes an equivalent solution with z_1 and z_2 interchanged. Allowing for this, defining

$$h = \sin u/\sin(\lambda - u), \qquad \gamma = (2\cos\lambda)^{-1} \tag{A6}$$

and using (5.34), we obtain

$$\begin{aligned}
\sum z_1^i z_2^j = \sum z_1^j z_2^i &= NL(L+1)/2, && i=j=0 \\
&= \frac{NL}{2}\left\{\binom{i-j}{i}\beta^{i+j}\gamma^{i-j} - \delta_{i,-j}\right\}, && j \leqslant 0 \leqslant i, \quad 0 < i-j \leqslant L \\
&= 0, && ij > 0
\end{aligned} \tag{A7}$$

(The restriction $i-j \leqslant L$ can probably be relaxed somewhat, but this result is sufficient for our purposes.)

The result (A7) gives the sum of $z_1^i z_2^j$ over all distinct allowed solutions of (5.16), i.e., over all eigenvalues of $T_q \hat{T}_q$. Note that there are $NL(L+1)/2$ such eigenvalues, in agreement with the dimension of the truncated $r=2$ matrices—see the remarks before Eq. (5.19).

From (2.11), $Z_2 = \sum (T_q \hat{T}_q)^M$, where the sum is over all the $NL(L+1)/2$ eigenvalues. Using the binomial theorem to expand the RHS of (5.18) in powers of z_1 and z_2, we obtain

$$Z_2 = \xi^{2M} \sum_{i,j} \binom{2M}{M-i}\binom{2M}{M-j} h^{i+j} \sum z_1^i z_2^j \tag{A8}$$

the i and j sums being from $-M$ to M.

The summand is symmetric in i and j. Using this and (A7), we obtain, for $2M \leqslant L$,

$$\begin{aligned}
Z_2 = \frac{NL\xi^{2M}}{2}\Bigg\{ & L\binom{2M}{M}^2 - \sum_{i=-M}^{M}\binom{2M}{M-i}^2 \\
& + 2\sum_{i=0}^{M}\sum_{j=-M}^{0}\binom{2M}{M-i}\binom{2M}{M-j}\binom{i-j}{i} h^{i+j}\gamma^{i-j}\Bigg\}
\end{aligned} \tag{A9}$$

which makes it clear that Z_2 is real when $0 < u < \lambda$.

We see that (for L sufficiently large), Z_2 is indeed a polynomial in L of degree 2, and from (5.35) the leading (quadratic) terms in $Z_1^2/2Z_0$, in agreement with (5.27).

The remaining terms are proportional to L. When M is large, they grow exponentially with M. Ignoring factors that grow (or decay) at less-than-exponential rates, the term preceding the double sum is proportional to $(4\xi)^{2M}$—as is the leading term in (A9).

The terms in the double (i, j) summation are all positive, and the sum is dominated by the maximum term, occurring when $i = Mf$ and $j = -Mg$, where

$$\begin{aligned} f(1+f)/\{(1-f)(f+g)\} &= \gamma h \\ g(1+g)/\{(1-g)(f+g)\} &= \gamma/h \end{aligned} \tag{A10}$$

For $N \geqslant 3$ and $0 < u_q < \pi/N$, it appears that these equations have a unique solution for f, g in the range $0 < f,\ g < 1$. The corresponding term in the summand is (for M large) $\{16/[(1-f^2)(1-g^2)]\}^M$. Hence the contribution of the double sum to Z_2 is

$$NL(4\xi)^{2M}/[(1-f^2)(1-g^2)]^M \tag{A11}$$

This dominates the contribution from the preceding term. Using (5.26), we deduce that the interfacial tension ε_2 is given by

$$(\varepsilon_2 - 2\varepsilon_1)/k_B\mathscr{T} = \ln[(1-f^2)(1-g^2)] \tag{A12}$$

The RHS of this expression is plainly negative, so $\varepsilon_2 < 2\varepsilon_1$. The RHS of the inequality (5.38) can be written as

$$-\ln\{(1+\gamma h))(1+\gamma/h)/4\} \tag{A13}$$

from which we can verify explicitly that it is satisfied by our result (A12).

A2. Comparison with Formulas of Section 5

For $r = 1$ we have the two results (5.36) and (5.45)–(5.46). We can verify that they are equivalent by solving (5.45) for $\exp(2i\chi_0)$, and obtaining

$$\begin{aligned} &\sec(\chi_0 + \lambda/2) : \sec(\chi_0 - \lambda/2) : \sec(\chi_0 + u - \lambda/2) \\ &\quad = 2\sin u : 2\sin(\lambda - u) : \sin\lambda \end{aligned} \tag{A14}$$

Using these ratios in (5.46), we get the earlier result (5.36).

For $r=2$ we should compare Eqs. (A10)–(A12) with (5.45)–(5.46). We find that the f, g herein are related to χ_0 by

$$1+f=\frac{2\sin u\cos\chi_0}{\sin\lambda\cos(\chi_0+u-\lambda)}$$

$$1-f=\frac{2\sin(\lambda-u)\cos(\chi_0-\lambda)}{\sin\lambda\cos(\chi_0+u-\lambda)}$$

$$1+g=\frac{2\sin(\lambda-u)\cos\chi_0}{\sin\lambda\cos(\chi_0+u)} \tag{A15}$$

$$1-g=\frac{2\sin u\cos(\chi_0+\lambda)}{\sin\lambda\cos(\chi_0+u)}$$

$$\frac{f}{g}=\frac{\cos(\chi_0-\lambda)}{\cos(\chi_0+\lambda)} \tag{A16}$$

Using these formulas in (A12) [together with (5.34)–(5.36)], we obtain the result (5.46).

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